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Suns, Moons, and Quasi-Polyhedra

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1. INTRODUCTION

The important concept of a *sun*, which was first introduced by Efimov and Stečkin in [17], arises quite naturally in the general theory of approximation in normed linear spaces. We recall that a set V is a sun iff whenever $v_0 \in V$ is a best approximation to some element x ($\notin V$), then v_0 is a best approximation to every element on the ray from v_0 through x. Since every convex set has this property, a sun may be regarded as a generalization of a convex set. Vlasov [21] showed that in a smooth Banach space every proximinal sun is convex. (A brief proof of this will be given in Section 2). Perhaps the most famous unsolved problem in approximation theory is whether or not every Tchebycheff set in a Hilbert space is convex. In view of Vlasov's result, this problem may be stated equivalently as "Is every Tchebycheff set in a Hilbert space a sun?" Brosowski [6] has shown that being a sun is equivalent to being a Kolmogorov set (cf. Theorem 2.4). Also, he and his colleagues have indicated a theory of approximation for such sets which closely parallels the known linear or convex theory (cf., e.g., [10]). In recent years, a number of writers have studied certain classes of suns (e.g., the so-called "regular sets" introduced by Brosowski [5]); these authors have tried to determine, among other things, those spaces in which every sun is a member of this class [3-7, 8, 10].

In the present work, we define the concept of a "moon"[1], which is a

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[1] Originally called "sign regular," [9]. The present name was given on "Moonday," July 21, 1969, for obvious reasons.

generalization of a sun. We are especially interested in determining those normed linear spaces in which every moon is a sun. Knowledge of such spaces is often quite useful in practice since it is generally much easier to verify that a given set is a moon than verify it is a sun. Our approach to this problem is via certain geometric properties of the points of the unit sphere, in particular being "nonlunar", "strongly nonlunar", or "quasi-polyhedral" (abbr. QP) (in order of decreasing generality).

Section 2 includes the basic definitions, notation, and a number of general results. The main result of that section (Theorem 2.18) states that if each point of the unit sphere is strongly nonlunar, then every moon is a sun. We observe (Theorem 2.22) that every point of the unit sphere is QP if the unit ball is a "convex polytope" in the sense of Maserick [19]. Further, the finite-dimensional spaces in which each point of the unit sphere is QP are precisely those whose unit ball is polyhedral (Theorem 2.19). In Section 3 we consider certain product spaces. We prove, for example (Theorem 3.2), that each point of the unit sphere of the c_0-product of normed spaces is strongly nonlunar (or QP) iff each of the component spaces has the same property. The space $C_0(T)$, T locally compact Hausdorff, is studied in Section 4. The main results there (Theorems 4.1 and 4.4) may be summarized as follows: Each point of the unit sphere in $C_0(T)$ is strongly nonlunar; each point is QP iff T is discrete. In Section 5 a similar study is made of the space $L_1(T, \Sigma, \mu)$, where (T, Σ, μ) is σ-finite. The main results there (Theorems 5.4 and 5.6) may be stated as: Each point of the unit sphere in $L_1(T, \Sigma, \mu)$ is strongly nonlunar iff T is purely atomic; each point is QP iff T is a finite union of atoms. In Section 6 we remark about certain related matters and pose some open problems. In particular, we observe a certain close relationship (Theorem 6.3) between the QP property, property (P) of Brown [12], and property Q of Deutsch and Lindahl [15].

2. Notation, Definitions, and Some General Results

Let X be a real normed linear space, X^* its dual space,

$$B(x, r) = \{y \in X : \| x - y \| < r\}, \text{ and } S(X) = \{x \in X : \| x \| = 1\}.$$

For any $x \in X$, we define the *peak set* of x by

$$P(x) = \{x^* \in S(X^*) : x^*(x) = \| x \|\}.$$

Given v_0, $x \in X$, we define the (open) *cone of support* at v_0 in the direction x, by

$$\begin{aligned} K(v_0, x) &= \{v \in X : x^*(v - v_0) < 0 \ \forall\ x^* \in P(v_0 - x)\} \\ &= \{v \in X : x^*(v - x) < \| v_0 - x \| \ \forall\ x^* \in P(v_0 - x)\}^2. \end{aligned}$$

[2] Observe that $K(v_0, x) = \{v \in X\colon\ x^*(v - v_0) > 0\ \ \forall x^* \in P(x - v_0)\}$, a fact which is sometimes useful.

Since $P(v_0 - x)$ is a weak* compact convex extremal subset of $S(X^*)$, we can restrict ourselves, in the definition of $K(v_0, x)$, to those $x^* \in \text{ext}\, P(v_0 - x)$. (Here, and in the sequel, "ext" is an abbreviation for "the set of extreme points of.") In dealing with more than one normed linear space, we shall often use subscripts to emphasize the space in which we consider the ball, cone, etc. ; e.g., $B_X(x, r)$, $K_X(v_0, x)$, etc.

There is a useful alternate representation for $K(v_0, x)$.

LEMMA 2.1. $K(v_0, x) = \bigcup_{\lambda > 0} B(v_0 + \lambda(x - v_0), \lambda \| v_0 - x \|)$.

Proof. If $\| v - v_0 - \lambda(x - v_0) \| < \lambda \| v_0 - x \|$, then for any $x^* \in P(v_0 - x)$,

$$\begin{aligned} \lambda \| v_0 - x \| &> x^*[v - v_0 - \lambda(x - v_0)] \\ &= x^*(v - v_0) + \lambda \| v_0 - x \| ; \end{aligned}$$

so $x^*(v - v_0) < 0$ and $v \in K(v_0, x)$.

Conversely, let $v \in K(v_0, x)$. The open line segment (v_0, v) must intersect $B(x, \| v_0 - x \|)$ for, otherwise, by the Eidelheit separation theorem, we could find an $x^* \in P(v_0 - x)$ with $x^*(v - v_0) \geqslant 0$, which contradicts the choice of v. Choose $0 < \lambda < 1$ such that $z = \lambda v_0 + (1 - \lambda) v$ satisfies $\| z - x \| < \| v_0 - x \|$. Taking $\alpha = 1/(1 - \lambda)$, we obtain

$$\| v - [v_0 + \alpha(x - v_0)] \| = \frac{1}{1 - \lambda} \| z - x \| < \alpha \| v_0 - x \|.$$

Thus $v \in B(v_0 + \alpha(x - v_0), \alpha \| v_0 - x \|)$. ∎

COROLLARY 2.2. *If* $x_1 = v_0 + \lambda(x - v_0)$ *for some* $\lambda > 0$, *then* $K(v_0, x_1) = K(v_0, x)$.

DEFINITIONS 2.3. A set $V \subset X$ is called a *Kolmogorov set* iff whenever $v_0 \in V$ is a best approximation to $x \in X$, then

$$\text{(K)} \qquad \min_{x^* \in P(x - v_0)} x^*(v - v_0) \leqslant 0 \qquad \forall\, v \in V.$$

The set V is called a *sun* iff whenever $v_0 \in V$ is a best approximation to $x \in X$, then v_0 is also a best approximation to $v_0 + \lambda(x - v_0)$ $\forall \lambda \geqslant 0$, i.e. (if $x \neq v_0$), to each point on the ray from v_0 through x.

An interesting exposition on Kolmogorov sets was given by Brosowski [8]. It is easy to show that the condition (K) is always *sufficient* for v_0 to be a best approximation to x. The necessity of condition (K) was recently discussed by Brosowski and Wegmann [10]. The concept of a sun was introduced by Efimov and Stečkin [17] and further developed by Vlasov [21] (cf. also the encyclopedic monograph of Singer [20]).

THEOREM 2.4. *Let $V \subset X$. The following are equivalent.*

(1) *V is a Kolmogorov set.*

(2) *$V \cap K(v_0, x) = \varnothing$ whenever $v_0 \in V$ is a best approximation to x.*

(3) *V is a sun.*

Proof. (1) $\Rightarrow$ (2). Let $v_0 \in V$ be a best approximation to x. By hypothesis,

$$\max_{x^* \in P(v_0 - x)} x^*(v - v_0) \geqslant 0 \qquad \forall v \in V.$$

On the other hand,

$$\begin{aligned} K(v_0, x) &= \{v: x^*(v - v_0) < 0 \ \forall x^* \in P(v_0 - x)\} \\ &= \{v: \max_{x^* \in P(v_0 - x)} x^*(v - v_0) < 0\}, \end{aligned}$$

and so $V \cap K(v_0, x) = \varnothing$.

(2) $\Rightarrow$ (3). Let $v_0 \in V$ be a best approximation to x and let $\lambda > 0$. If $x_1 = v_0 + \lambda(x - v_0)$ then $K(v_0, x_1) = K(v_0, x)$ by Corollary 2.2 and so $K(v_0, x_1) \cap V = \varnothing$. From Lemma 2.1 we obtain, in particular, that

$$V \cap B(x_1, \| x_1 - v_0 \|) = \varnothing,$$

and so v_0 is a best approximation to x_1.

(3) $\Rightarrow$ (1). Let $v_0 \in V$ be a best approximation to x and let $v \in V$. If $x^*(v - v_0) > 0 \ \forall x^* \in P(x - v_0)$, then $v \in K(v_0, x)$, and so

$$v \in B(v_0 + \lambda(x - v_0), \lambda \| x - v_0 \|)$$

for some $\lambda > 0$. Thus

$$\| v_0 + \lambda(x - v_0) - v \| < \lambda \| x - v_0 \| = \| v_0 + \lambda(x - v_0) - v_0 \|,$$

which contradicts the hypothesis that V is a sun. Hence

$$\min_{x^* \in P(x - v_0)} x^*(v - v_0) \leqslant 0. \quad \blacksquare$$

The equivalence of (1) and (3) in Theorem 2.4 had been proved earlier by Brosowski [6] by a different method.

A normed linear space X is called *smooth* if there is a unique supporting hyperplane to the unit sphere at each point, i.e., if $P(v_0)$ is a singleton for each $v_0 \in S(X)$. A subset V of X is called *proximinal* if each $x \in X$ has at least one best approximation in V. We can now give a new short proof of a well-known result of Vlasov (cf. [21; or 20, p. 344]).

THEOREM 2.5. *Let X be a smooth normed linear space. Then each proximinal sun is convex.*

Proof. Let V be a proximinal sun. If V is not convex, there exist $v_1, v_2 \in V$ such that $x = \lambda v_1 + (1 - \lambda) v_2 \notin V$ for some $0 < \lambda < 1$. Let $v_0 \in V$ be a best approximation to x. Let $\{x^*\} = P(v_0 - x)$. By Theorem 2.4, $V \cap K(v_0, x) = \varnothing$, and so $x^*(v_i - v_0) \geqslant 0$ for $i = 1, 2$. Thus

$$0 < \| v_0 - x \| = x^*(v_0 - x) = \lambda x^*(v_0 - v_1) + (1 - \lambda) x^*(v_0 - v_2) \leqslant 0,$$

a contradiction. ▌

Using Theorem 2.4, one can also easily verify the known fact that every convex set is a sun. (It is easy to construct examples of nonconvex suns.)

Theorem 2.4 suggests (at least) one way of generalizing the concept of a sun.

DEFINITION 2.6. Let $V \subset X$. A point $v_0 \in V$ is called a *lunar point* if $x \in X$ and $V \cap K(v_0, x) \neq \varnothing$ imply $v_0 \in \overline{V \cap K(v_0, x)}$. (As a consequence of the next lemma, we may assume in this definition that x has v_0 as a best approximation from V.) V is called a *moon* if each of its points is lunar.

LEMMA 2.7. *Let $V \subset X$ and $v_0 \in V$. The following are equivalent*:

(1) *v_0 is a lunar point.*

(2) *Whenever v_0 is a best approximation to an $x \in X$ with*

$$V \cap K(v_0, x) \neq \varnothing, \text{ then } v_0 \in \overline{V \cap K(v_0, x)}.$$

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let $x \in X$ and $V \cap K(v_0, x) \neq \varnothing$. We have to show

$$v_0 \in \overline{V \cap K(v_0, x)}.$$

If v_0 is not a local best approximation to x (i.e., if $\forall \epsilon > 0$ there is a $v_\epsilon \in V$ such that $\| v_\epsilon - v_0 \| < \epsilon$ and $\| v_\epsilon - x \| < \| v_0 - x \|$), then

$$v_\epsilon \in B(x, \| v_0 - x \|) \subset K(v_0, x),$$

so $v_0 \in \overline{V \cap K(v_0, x)}$. Thus we can assume v_0 is a best approximation to x from $V \cap B(v_0, \epsilon)$ for some $\epsilon > 0$. Let $y = v_0 + \lambda(x - v_0)$ where

$$0 < \lambda < \epsilon/2 \| v_0 - x \| .$$

Then $K(v_0, y) = K(v_0, x)$, $\| y - v_0 \| < \epsilon/2$, and v_0 is a best approximation to y from V. Thus $v_0 \in \overline{V \cap K(v_0, y)} = \overline{V \cap K(v_0, x)}$. ▌

COROLLARY 2.8. *Every sun is a moon.*

This follows from Theorem 2.4 and Lemma 2.7.

In the important special case $V = S(X)$, the definition of a lunar point of V can be somewhat simplified. Indeed, $v_0 \in S(X)$ is a lunar point iff for each $x \in B(0, 1)$ having v_0 as a best approximation from $S(X)$, $v_0 \in \overline{K(v_0, x) \cap S(X)}$. To shorten the writing, we define, for each $v_0 \in S(X)$,

$$\begin{aligned} \mathcal{O}\!\ell(v_0) &= \{x \in B(0, 1) : v_0 \text{ is a best approximation to } x \text{ from } S(X)\} \\ &= \{x \in B(0, 1) : \| v_0 - x \| = 1 - \| x \|\} \\ &= \{x \in B(0, 1) : v_0 = x + (1 - \| x \|)\, u \text{ for some } u \in S(X)\}. \end{aligned}$$

Thus $v_0 \in S(X)$ is a lunar point iff $v_0 \in \overline{K(v_0, x) \cap S(X)}$ $\forall x \in \mathcal{O}\!\ell(v_0)$.

DEFINITIONS 2.9. Let $v_0 \in S(X)$.

(a) v_0 is called a nonlunar point of $S(X)$ if it is not a lunar point, i.e., if there is some $x \in B(0, 1)$ such that $v_0 \notin \overline{K(v_0, x) \cap S(X)}$.

(b) v_0 is called a *strongly nonlunar* point of $S(X)$ if for each $u \in K(v_0, 0)$ there is an $x \in B(0, 1)$ such that $u \in K(v_0, x)$ and $v_0 \notin \overline{K(v_0, x) \cap S(X)}$. The space X is called *strongly nonlunar* if each $v_0 \in S(X)$ is strongly nonlunar.

(c) v_0 is called a *quasi-polyhedral* (abbr. *QP*) point of $S(X)$ if

$$v_0 \notin \overline{K(v_0, 0) \cap S(X)}.$$

X is called a *QP-space* if each $v_0 \in S(X)$ is *QP*.

It should be noted that (by an argument similar to that used in the proof of Lemma 2.7) *the* $x \in B(0, 1)$ *which appears in the definitions of nonlunar and strongly nonlunar points may be restricted to lie in* $\mathcal{O}\!\ell(v_0)$. We leave to the reader the straightforward task of verifying that *the QP property is hereditary* (i.e., if X is *QP*, so is every subspace of X). On the other hand, Theorem 4.1 shows that strong nonlunarity is *not* a hereditary property.

In verifying whether a given point is nonlunar, strongly nonlunar, or *QP*, it is useful to observe that if $v_0 \in S(X)$ and $x \in B(0, 1)$, the following two conditions are equivalent:

(1) $v_0 \notin \overline{K(v_0, x) \cap S(X)}$.

(2) There exists an $\epsilon > 0$ such that $B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$.

THEOREM 2.10. *Let* $v_0 \in S(X)$ *and consider the following three statements*:

(1) v_0 *is QP*.

(2) v_0 *is strongly nonlunar*.

(3) v_0 *is nonlunar*.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

In addition, if any one of the following three conditions holds, then (3) $\Rightarrow$ (1), *and so, all three statements above are equivalent.*

(a) $v_0 \in \operatorname{ext} S(X)$.

(b) *X is two-dimensional.*

(c) *X is smooth.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial (e.g., for (1) $\Rightarrow$ (2), take $x = 0$). Now suppose v_0 is nonlunar. Then there is an $x \in \mathcal{O}\!\ell(v_0)$ such that $v_0 \notin \overline{K(v_0, x) \cap S(X)}$. We shall show that if any one of the conditions (a), (b), or (c) is satisfied, then v_0 is QP. This will be the case, in particular, when $K(v_0, x) = K(v_0, 0)$.

Case 1. $v_0 \in \operatorname{ext} S(X)$.

Since $v_0 = x + (1 - \|x\|)\, u$ for some $u \in S(X)$, it follows that either $x = 0$ or $v_0 = \|x\|\, x/\|x\| + (1 - \|x\|)\, u$. If the latter is true, then $x/\|x\| = u = v_0$. Hence $x = \|x\|\, v_0$ and, in particular, $K(v_0, x) = K(v_0, 0)$.

Case 2. X is two-dimensional.

We may assume $v_0 \notin \operatorname{ext} S(X)$. Then v_0 must be interior to some line segment $L(v_0)$ in $S(X)$. In particular, v_0 is a smooth point,

$$K(v_0, 0) \cap S(X) \subset S(X) \sim L(v_0),$$

and so, $v_0 \notin \overline{K(v_0, 0) \cap S(X)}$, i.e., v_0 is QP.

Case 3. X is smooth.

The proof in this case, and hence the theorem, will follow immediately from (3) of the following lemma.

LEMMA 2.11. *Let $v_0 \in S(X)$ and $x \in \mathcal{O}\!\ell(v_0)$. Then*:

(1) $P(v_0) = P(v_0 - x) \cap P(x)$.

(2) $K(v_0, x) \subset K(v_0, 0)$.

(3) *If X is smooth,* $K(v_0, x) = K(v_0, 0)$.

Proof of the Lemma. (1) Let $x^* \in P(v_0)$. Then

$$\begin{aligned}\|v_0 - x\| + \|x\| = \|v_0\| = x^*(v_0) &= x^*(v_0 - x) + x^*(x)\\ &\leqslant \|v_0 - x\| + \|x\|\end{aligned}$$

and so, $x^* \in P(v_0 - x) \cap P(x)$. Conversely, suppose $x^* \in P(v_0 - x) \cap P(x)$. Then

$$x^*(v_0) = x^*(v_0 - x) + x^*(x) = \|v_0 - x\| + \|x\| = 1,$$

so $x^* \in P(v_0)$.

(2) From (1) we obtain $P(v_0) \subset P(v_0 - x)$ and so, $K(v_0, x) \subset K(v_0, 0)$.

(3) If X is smooth, then $P(y)$ is a singleton for each $0 \neq y \in X$; so by (1) we obtain $P(v_0) = P(v_0 - x)$ and hence $K(v_0, 0) = K(v_0, x)$.

This proves Lemma 2.11 and hence completes the proof of Theorem 2.10. ▮

Remark 2.12. For a two-dimensional space, we have shown that the concepts "nonlunar," "strongly nonlunar," and "QP" are the same. There exists, however, a three-dimensional space which contains nonlunar points which are not strongly nonlunar [11]. Also, we shall see later that there are infinite-dimensional strongly nonlunar spaces which are not QP. However, it is an open question whether there are *finite*-dimensional spaces with this property.

During the course of the proof of Case 2 in Theorem 2.10, we have actually verified the following result:

LEMMA 2.13. *Let X be two-dimensional and $v_0 \in S(X)$. If v_0 is lunar, then* $v_0 \in \operatorname{ext} S(X)$.

It is clear that $S(X)$—or, for that matter, any symmetric subset of $S(X)$—is *never* a sun. On the other hand, with the aid of Theorem 2.10, we can give certain conditions which insure that $S(X)$ is a moon.

THEOREM 2.14. *If X is strictly convex, then $S(X)$ is a moon.*

Proof. Let $v_0 \in S(X)$ and $x \in \mathcal{O}(v_0)$. By the strict convexity, $x = \|x\| v_0$, and so $K(v_0, x) = K(v_0, 0)$. Since each $x^* \in P(v_0)$ attains its norm on $S(X)$ only at v_0, it follows that $x^*(v) < 1 = x^*(v_0)$ $\forall v \in S(X) \sim \{v_0\}$. Thus

$$K(v_0, x) \cap S(X) = K(v_0, 0) \cap S(X) = S(X) \sim \{v_0\},$$

and so $v_0 \in \overline{K(v_0, x) \cap S(X)}$, i.e., v_0 is a lunar point. ▮

By combining Lemma 2.13 and Theorem 2.14, we obtain

COROLLARY 2.15. *Let X be two-dimensional. Then $S(X)$ is a moon if and only if X is strictly convex.*

A set $E \subset S(X)$ is called an *exposed set* of $S(X)$ if E is the intersection of $S(X)$ with a supporting hyperplane to $S(X)$, i.e., if $E = \{v \in S(X) : x^*(v) = 1\}$ for some $x^* \in S(X^*)$.

THEOREM 2.16. *Let X be smooth. Then $S(X)$ is a moon if and only if each exposed set of $S(X)$ has an empty interior relative to $S(X)$.*

Proof. If some exposed set E had a relative interior point v_0, then (by smoothness)

$$K(v_0, 0) \cap S(X) = S(X) \sim E,$$

and so $v_0 \notin \overline{K(v_0, 0) \cap S(X)}$. Thus $S(X)$ is not a moon.

Conversely, suppose each exposed set has an empty relative interior. Let $v_0 \in S(X)$ and $x \in \mathcal{Q}(v_0)$. Since X is smooth, $P(v_0 - x) = \{x^*\}$ is a singleton, so that $E = \{v \in S(X) : x^*(v) = 1\}$ is an exposed set which contains v_0. Note that $K(v_0, x) \cap S(X) = S(X) \sim E \neq \varnothing$. Since E has an empty relative interior, it follows that in each neighborhood of v_0 there points of $S(X) \sim E$. Thus $v_0 \in \overline{K(v_0, x) \cap S(X)}$, and so $S(X)$ is a moon. ▮

Remark 2.17. The theorem is not true without the smoothness assumption. A 3-space whose unit ball is a "double ice-cream cone" (i.e., the convex hull of the union of a circle and a line segment through its center, normal to its plane) provides an example. In this case, the vertices (in particular) are nonlunar points, but each exposed set of $S(X)$ has an empty relative interior.

The fundamental result concerning strong nonlunarity is the following.

THEOREM 2.18. *Let X be strongly nonlunar. A subset of X is a moon if and only if it is a sun.*

Proof. Every sun is a moon (Corollary 2.8). Let V be a moon which is not a sun. Then there is a $v_0 \in V$ which is a best approximation to some $x \in X$ with $K(v_0, x) \cap V \neq \varnothing$. Let $v \in K(v_0, x) \cap V$. By the strong nonlunarity of the sphere $S(x, \| v_0 - x \|)(= \| v_0 - x \| S(X) + x)$ at v_0, there exists an $x_1 \in B(x, \| v_0 - x \|)$ having v_0 as a best approximation in $S(x, \| v_0 - x \|)$ such that $v \in K(v_0, x_1)$ and $v_0 \notin \overline{K(v_0, x_1) \cap S(x, \| v_0 - x \|)}$, i.e., there is an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, x_1) \subset B(x, \| v_0 - x \|) \subset X \sim V,$$

and so $v_0 \notin \overline{K(v_0, x_1) \cap V}$. But this contradicts the fact that V is a moon. ▮

It is an open question whether the converse is true. That is, if every moon in X is a sun, must X be strongly nonlunar?

Before we characterize the finite-dimensional QP spaces, let us observe that $v_0 \in S(X)$ is QP iff there is an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap \overline{K(v_0, 0)} = B(v_0, \epsilon) \cap \overline{B(0, 1)}$$

which holds iff there is an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap \text{bd}\, K(v_0, 0) = B(v_0, \epsilon) \cap S(X).$$

(Here bd $K(v_0, 0)$ denotes the *boundary* of $K(v_0, 0)$.)

THEOREM 2.19. *A finite-dimensional space is QP if and only if its (closed) unit ball is a polytope.*

Proof. Let $\overline{B(0, 1)}$ be a polytope and let $v_0 \in S(X)$. Then $\overline{B(0, 1)} = \bigcap_{i \in I} E_i$, where I is finite, $E_i = \{x \in X : x_i^*(x) \leqslant 1\}$, and $x_i^* \in S(X^*)$. The hyperplanes $x_i^{*-1}(1)$ which determine the half-spaces E_i will be denoted by H_i. Let $I_0 = \{i \in I : v_0 \in H_i\}$ and set $\epsilon = \operatorname{dist}(v_0, \bigcup_{i \notin I_0} H_i)$. Since $\bigcup_{i \notin I_0} H_i$ is closed, $\epsilon > 0$. Now $\operatorname{dist}(v_0, H_i) = 1 - x_i^*(v_0)$ for every i (cf., e.g., [14, Lemma 2.1]), so that $\epsilon = \inf_{i \notin I_0} \operatorname{dist}(v_0, H_i) = \inf_{i \notin I_0} [1 - x_i^*(v_0)]$. We shall show that

$$B(v_0, \epsilon) \cap \Big(\bigcap_{i \in I} E_i\Big) = B(v_0, \epsilon) \cap \Big(\bigcap_{i \in I_0} E_i\Big). \tag{1}$$

Indeed, if (1) is false, there is an $x \in X$ with $x \in B(v_0, \epsilon)$ and $x \notin E_{i_0}$ for some $i_0 \in I \sim I_0$. Then

$$x_{i_0}^*(x) > 1 \geqslant x_{i_0}^*(v_0) + \epsilon,$$

and hence $\| x - v_0 \| > \epsilon$. This contradiction establishes (1). From (1) we obtain

$$\begin{aligned} B(v_0, \epsilon) \cap \overline{B(0, 1)} &= B(v_0, \epsilon) \cap \Big(\bigcap_{i \in I} E_i\Big) \\ &= B(v_0, \epsilon) \cap \Big(\bigcap_{i \in I_0} E_i\Big) = B(v_0, \epsilon) \cap \overline{K(v_0, 0)}. \end{aligned}$$

Thus v_0 is QP.

Conversely, suppose X is an n-dimensional QP space. Consider first the case $n = 2$. For each $v \in S(X)$, there is an $\epsilon_v > 0$ such that

$$B(v, \epsilon_v) \cap \operatorname{bd} K(v, 0) = B(v, \epsilon_v) \cap S(X). \tag{2}$$

By the compactness of $S(X)$, there is a finite set of $v_i \in S(X)$ such that $\{B(v_i, \epsilon_{v_1}\}_1^m$ covers $S(X)$. Hence

$$S(X) = \bigcup_1^m [B(v_i, \epsilon_{v_i}) \cap \operatorname{bd} K(v_i, 0)].$$

But since $\operatorname{bd} K(v, 0)$ consists of at most two lines for each $v \in S(X)$, it follows that $S(X)$ consists of a finite number of line segments, i.e., X is polyhedral. Now suppose $n > 2$. Then since the QP property is hereditary, each 2-dimensional subspace of X is QP. By the above argument, each 2-dimensional subspace of X is polyhedral. By a well-known result [18a, Theorem 4.7], it follows that X must be polyhedral. ▮

COROLLARY 2.20. *Let X be two-dimensional. Then each point of $S(X)$ is nonlunar if and only if $S(X)$ is a polygon.*

As an application of this corollary we consider the following unit sphere in the plane which has exactly two lunar points, the remaining being QP points.

EXAMPLE 2.21. *An "infinite polygon" in the plane.* Let $t_n = 1 - (1/2^n)$ $(n = 0, 1, 2,...)$ and define a function f on $[0, 1]$ to be linear on each subinterval $[t_n, t_{n+1}]$ and to satisfy $f(t_0) = f(0) = 1, f(t_{n+1}) = \frac{1}{2}(t_n + f(t_n))$ $(n = 0, 1,...)$, and $f(1) = 0$. Define $g(t) \equiv t - 1$ $(0 \leqslant t \leqslant 1)$,

$$f_1(t) \equiv -f(-t) \; (-1 \leqslant t \leqslant 0),$$

and $g_1(t) \equiv -g(-t)$ $(-1 \leqslant t \leqslant 0)$. Then the union $S(X)$ of the graphs of f, g, f_1, and g_1 is what we call an "infinite polygon" in the plane. Clearly, $S(X)$ is QP at every point, with the exception of the two "infinite" points $(1, 0), (-1, 0)$, and these must be lunar points.

Maserick [19] has defined a "convex polytope" P as an intersection of a family of half-spaces: $P = \bigcap_{i \in I} E_i$ (corresponding to the hyperplanes $\{H_i : i \in I\}$), such that, for every $x \in X$, there is a finite subcollection $I_0 \subset I$ with $x \in \bigcap_{i \notin I_0} E_i$.

THEOREM 2.22. *If $\overline{B(0, 1)}$ is a convex polytope (in the sense of* [19]), *then X is a QP space.*

Proof. Properties 2.3, 2.4, and 2.5 of [19] assert that, if $\overline{B(0, 1)}$ is a convex polytope and $v_0 \in S(X)$, then $I_0 = \{i \in I : v_0 \in H_i\}$ is a nonempty finite family and $\bigcup_{i \notin I_0} H_i$ is a closed set. Setting $\epsilon = \operatorname{dist}(v_0, \bigcup_{i \notin I_0} H_i)$, we observe that exactly the same argument used in the proof of Theorem 2.19 shows that v_0 is QP. ▌

From the results of [19], we quote the following:

(1) Convex polytopes in infinite-dimensional spaces have no extreme points.

(2) If the unit ball of X is a convex polytope, so is the unit ball of every subspace of X.

(3) The unit ball of $c_0(T)$ is a convex polytope for every discrete T;

(4) If the unit ball of X is a convex polytope with a countable number of exposed sets, then X is isometric to a subspace of c_0.

We give now an example, which is a simplification of a more general one given in [19], of a QP space whose unit ball is *not* a convex polytope. Let X be the l_1-product of the real line R and c_0, i.e., $X = (R \times c_0)_{l_1(2)}$. (See Section 3 for some basic results on product spaces.) Then X is $\underline{QP}$ since both R and c_0 are (Theorem 3.5), but, by property 2.4 of [19], $\overline{B(0, 1)}$ is not a convex polytope since the vertex $x = (1, 0, 0, \ldots)$ belongs to infinitely many exposed sets.

3. Product Spaces

Let I be an index set and let Y be a normed linear space of real-valued functions on I. If, for each $i \in I$, a normed linear space X_i is given, $(\prod_{i \in I} X_i)_Y$ denotes the (Y-product) space of all functions x on I such that

(1) $x(i) \in X_i$ for every $i \in I$,

(2) If ν_x is the function on I defined by $\nu_x(i) = \| x(i) \|$, then $\nu_x \in Y$.

We define a norm on $(\prod_{i \in I} X_i)_Y$ by $\| x \| = \| \nu_x \|_Y$.

We shall be mainly interested in the cases where $Y = c_0(I)$, $l_1(I)$, or $l_\infty(I)$. It is well known (cf. e.g., [13, p. 31]) that the dual space

$$\Big(\prod_{i \in I} X_i\Big)^*_{c_0(I)} \quad \Big[\text{resp. } \Big(\prod_{i \in I} X_i\Big)^*_{l_1(I)}\Big]$$

may be identified with

$$\Big(\prod_{i \in I} X_i^*\Big)_{l_1(I)} \quad \Big[\text{resp. } \Big(\prod_{i \in I} X_i^*\Big)_{l_\infty(I)}\Big]$$

via the mapping $x^* \to (x^*(i))_{i \in I}$, with $x^*(i) \in X_i^*$, defined by

$$x^*(x) = \sum_{i \in I} x^*(i)\, x(i)$$

for every x in the product space. If $X = (\prod_{i \in I} X_i)_{l_1(I)}$, then $x \in \operatorname{ext} S(X)$ if and only if $x(i) \in \operatorname{ext} S(X_i)$ for some $i = i_0$, and $x(i) = 0$ if $i \neq i_0$. If $X = (\prod_{i \in I} X_i)_{l_\infty(I)}$, then $x \in \operatorname{ext} S(X)$ if and only if $x(i) \in \operatorname{ext} S(X_i)$ for every $i \in I$.

We first consider the space $X = (\prod_{i \in I} X_i)_{c_0(I)}$. Let $x \in X$. We define the critical set of x by

$$\operatorname{crit} x = \{i \in I : \| x(i) \| = \| x \|\}.$$

Observe that if $v_0 \in S(X)$ and $x \in \mathcal{O}\!\ell(v_0)$, then $\operatorname{crit} v_0 \subset \operatorname{crit}(v_0 - x)$ (since if $i \in \operatorname{crit} v_0$, then

$$\| v_0 \| = \| v_0(i) \| \leqslant \| v_0(i) - x(i) \| + \| x(i) \| \leqslant \| v_0 - x \| + \| x \| = \| v_0 \|,$$

and so $\| v_0(i) - x(i) \| = \| v_0 - x \|$).

For any v_0, $x \in (\prod_{i\in I} X_i)_{c_0(I)}$ we have

$$\begin{aligned} K(v_0, x) &= \{v\colon \text{For each } i \in \operatorname{crit}(v_0 - x),\ x^*(i)[v(i) - v_0(i)] < 0 \\ &\qquad \text{for every } x^*(i) \in (\text{ext})\ P[v_0(i) - x(i)]\} \\ &= \{v\colon \text{For each } i \in \operatorname{crit}(v_0 - x),\ v(i) \in K_{X_i}(v_0(i), x(i))\}. \end{aligned}$$

LEMMA 3.1. *Let* $X = (\prod_{i\in I} X_i)_{c_0(I)}$ *and* $v_0 \in S(X)$. *Then* v_0 *is strongly nonlunar (resp. QP) if and only if, for every* $i \in \operatorname{crit} v_0$, $v_0(i)$ *is strongly nonlunar* (resp. *QP*) *in* $S(X_i)$.

Proof. Let v_0 be strongly nonlunar and let $i_0 \in \operatorname{crit} v_0$. We show $v_0(i_0)$ is strongly nonlunar in $S(X_{i_0})$. Let $u(i_0) \in K_{X_{i_0}}(v_0(i_0), 0)$. Define u by

$$u(i) = \begin{cases} u(i_0), & \text{if } i = i_0, \\ 0 & \text{if } i \neq i_0. \end{cases}$$

Then $u \in K(v_0, 0)$. Thus by strong nonlunarity there exists an $x \in \mathscr{A}(v_0)$. $\| x \| < 1$, such that $u \in K(v_0, x)$, and there exists an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1).$$

In particular, $x(i) \in B_{X_i}(0, 1)$ for every i. Now if $\| v - v_0 \| < \epsilon$ and if for every $i \in \operatorname{crit}(v_0 - x)$, $v(i) \in K_{X_i}(v_0(i), x(i))$, then $\| v \| < 1$. Since $u \in K(v_0, x)$ and $i_0 \in \operatorname{crit} v_0 \subset \operatorname{crit}(v_0 - x)$, it follows that $u(i_0) \in K_{X_{i_0}}(v_0(i_0), x(i_0))$. Also, if $\| v(i_0) - v_0(i_0)\| < \epsilon$ and $v(i_0) \in K_{X_{i_0}}(v_0(i_0), x(i_0))$, define

$$v(i) = \begin{cases} v(i_0), & \text{if } i = i_0, \\ \left(1 - \frac{\epsilon}{2}\right) v_0(i) + \frac{\epsilon}{2} x(i), & \text{if } i \neq i_0. \end{cases}$$

Then $\| v - v_0 \| < \epsilon$ and $v \in K(v_0, x)$, so that $\| v \| < 1$. In particular, $\| v(i_0)\| < 1$. This shows that $v_0(i_0)$ is strongly nonlunar.

Conversely, suppose that for each $i \in \operatorname{crit} v_0$, $v_0(i)$ is strongly nonlunar. Thus, for every $i \in \operatorname{crit} v_0$, if $u(i) \in K_{X_i}(v_0(i), 0)$, there exist $y(i) \in B_{X_i}(0, 1)$ and $\epsilon(i) > 0$ such that $u(i) \in K_{X_i}(v_0(i), y(i))$, and if $\| v(i) - v_0(i)\| < \epsilon(i)$ and $v(i) \in K_{X_i}(v_0(i), y(i))$, then $\| v(i)\| < 1$. We may assume that $\| v_0(i) - y(i)\|$ is constant for $i \in \operatorname{crit} v_0$. Now

$$\sup_{i \notin \operatorname{crit} v_0} \| v_0(i)\| = 1 - \delta \qquad \text{for some} \quad \delta > 0.$$

Let $\epsilon = \min\{\delta, \min_{i\in\operatorname{crit} v_0} \epsilon(i)\}$ and define x by

$$x(i) = \begin{cases} y(i), & \text{if } i \in \operatorname{crit} v_0, \\ v_0(i), & \text{if } i \notin \operatorname{crit} v_0. \end{cases}$$

Then $\| x \| < 1$ and $\text{crit}(v_0 - x) = \text{crit } v_0$. Let $u \in K(v_0, 0)$. Then for every $i \in \text{crit } v_0$, we have $u(i) \in K_{X_i}(v_0(i), 0)$, and so

$$u(i) \in K_{X_i}(v_0(i), y(i)) = K_{X_i}(v_0(i), x(i)),$$

i.e., $u \in K(v_0, x)$. If $\| v - v_0 \| < \epsilon$ and $v \in K(v_0, x)$, then $\| v(i) - v_0(i) \| < \epsilon$ for every i, and for each $i \in \text{crit } v_0$, $v(i) \in K_{X_i}(v_0(i), x(i))$ and so $\| v(i) \| < 1$. If $i \notin \text{crit } v_0$, then $\| v(i) \| < \| v_0(i) \| + \epsilon \leqslant 1 - \delta + \epsilon \leqslant 1$. Thus $\| v \| < 1$. We have shown

$$B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1),$$

and so v_0 is strongly nonlunar.

The proof of the analogous result with the QP condition is similar, but simpler. ∎

As an easy consequence of this lemma we obtain

THEOREM 3.2. *Let $X = (\prod_{i \in I} X_i)_{c_0(I)}$. Then X is strongly nonlunar (resp. QP) if and only if each X_i is strongly nonlunar (resp. QP).*

COROLLARY 3.3. *For any index set T, the space $c_0(T)$ is QP.*

We turn next to the l_1-product of a finite number of normed linear spaces. Let $X = (\prod_{i \in I} X_i)_{l_1(I)}$, and $v_0, x \in X$. Then

$$K(v_0, x) = \Big\{ v \in X \colon \sum_{i \in I} x^*(i)[v(i) - v_0(i)] < 0 \quad \text{whenever} \quad x^*(i) \in (\text{ext})\, P[v_0(i) - x(i)]. \Big\}$$

LEMMA 3.4. *Let $X = (X_1 \times X_2)_{l_1(I)}$, where $I = \{1, 2\}$, and let $v_0 \in S(X)$. If $v_0(i)/\| v_0(i) \|$ is QP in $S(X_i)$ whenever $v_0(i) \neq 0$, then v_0 is QP in $S(X)$.*

Proof. Assume first that both $v_0(1)$ and $v_0(2)$ are $\neq 0$. By assumption, we can choose an ϵ,

$$0 < \epsilon < \min \left(\frac{\| v_0(1) \|}{2}, \frac{\| v_0(2) \|}{2} \right)$$

such that for $i = 1, 2$,

$$B_{X_i}\left(\frac{v_0(i)}{\| v_0(i) \|}, \frac{4\epsilon}{\| v_0(i) \|} \right) \cap K_{X_i}\left(\frac{v_0(i)}{\| v_0(i) \|}, 0 \right) \subset B_{X_i}(0, 1).$$

Now let $v \in K(v_0, 0) \cap B(v_0, \epsilon)$. Thus $\| v(1) - v_0(1)\| + \| v(2) - v_0(2)\| < \epsilon$ and

$$\max_{x^*(1) \in P[v_0(1)]} x^*(1)[v(1) - v_0(1)] + \max_{x^*(2) \in P[v_0(2)]} x^*(2)[v(2) - v_0(2)] < 0.$$

There is a scalar α such that

$$\max_{x^*(1) \in P[v_0(1)]} x^*(1)[v(1) - v_0(1)] < \alpha,$$

$$\max_{x^*(2) \in P[v_0(2)]} x^*(2)[v(2) - v_0(2)] < -\alpha.$$

Since $v \in B(v_0, \epsilon)$, it follows that $|\alpha| < \epsilon$. If $x^*(1) \in P[v_0(1)]$, then

$$x^*(1)\left[\frac{v(1)}{\| v_0(1)\| + \alpha} - \frac{v_0(1)}{\| v_0(1)\|}\right]$$

$$= \frac{\| v_0(1)\|}{\| v_0(1)\| (\| v_0(1)\| + \alpha)} \{x^*(1)[v(1) - v_0(1)] - \alpha\} < 0$$

so

$$\frac{v(1)}{\| v_0(1)\| + \alpha} \in K_{X_1}\left(\frac{v_0(1)}{\| v_0(1)\|}, 0\right).$$

Moreover,

$$\left\| \frac{v(1)}{\| v_0(1)\| + \alpha} - \frac{v_0(1)}{\| v_0(1)\|} \right\| \leqslant \frac{\| v(1) - v_0(1)\|}{\| v_0(1)\| + \alpha} + \frac{|\alpha|}{\| v_0(1)\| + \alpha}$$

$$< \frac{\epsilon}{\| v_0(1)\| + \alpha} + \frac{\epsilon}{\| v_0(1)\| + \alpha} < \frac{4\epsilon}{\| v_0(1)\|},$$

so that, by (1), we have

$$\frac{\| v(1)\|}{\| v_0(1)\| + \alpha} < 1 \qquad \text{or} \qquad \| v(1)\| < \| v_0(1)\| + \alpha.$$

Similarly, we get $\| v(2)\| < \| v_0(2)\| - \alpha$. Hence

$$\| v \| = \| v(1)\| + \| v(2)\| < \| v_0(1)\| + \| v_0(2)\| = 1.$$

Thus $B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1)$ and so v_0 is QP. In the case when $v_0(1) = 0$ or $v_0(2) = 0$, the proof is similar but simpler. ∎

By induction, we obtain

LEMMA 3.5. *Let $X = (\prod_{i \in I} X_i)_{l_1(I)}$, where I is finite and $v_0 \in S(X)$. If $v_0(i)/\| v_0(i)\|$ is QP in $S(X_i)$ whenever $v_0(i) \neq 0$, then v_0 is QP in $S(X)$.*

As an immediate consequence of this lemma, we have

THEOREM 3.6. *Let* $X = (\prod_{i \in I} X_i)_{l_1(I)}$, *where* I *is finite. If each of the spaces* X_i *is* QP, *then* X *is* QP.

Remark 3.7. An analogous result for I infinite is not valid since (by Theorem 5.6 in the sequel) l_1 is not a QP-space.

4. THE SPACE $C_0(T)$

Throughout this section T will denote a locally compact Hausdorff space and $X = C_0(T)$—the space of real-valued continuous functions on T, vanishing at infinity, endowed with the uniform norm [18; p. 86]. Thus $x \in X$ iff x is continuous and, for each $\epsilon > 0$, the set $\{t \in T : |x(t)| \geqslant \epsilon\}$ is compact. Since the extreme points of $S(X^*)$ are just (plus or minus) the "point evaluations", we may identify ext $P(x)$ with

$$\text{crit}\, x \equiv \text{crit}\, x^+ \cup \text{crit}\, x^-,$$

where

$$\text{crit}\, x^{\pm} = \{t \in T : x(t) = \pm \| x \|\}.$$

Hence, for any $v_0, x \in X$, we have

$$\begin{aligned} K(v_0, x) = \{v \in X : v(t) < v_0(t) \quad & \text{if} \quad t \in \text{crit}(v_0 - x)^+, \\ v(t) > v_0(t) \quad & \text{if} \quad t \in \text{crit}(v_0 - x)^-\}. \end{aligned}$$

THEOREM 4.1. $C_0(T)$ *is strongly nonlunar.*

Proof. Let $v_0 \in S(X)$ and $v_1 \in K(v_0, 0)$. Choose $0 < \delta < 1$ such that

$$\delta < \min\{| v_0(t) - v_1(t)| : t \in \text{crit}\, v_0\},$$

and set

$$K^+ = \left\{t: v_0(t) \geqslant 1 - \frac{\delta}{3} > 1 - \frac{2\delta}{3} \geqslant v_1(t)\right\},$$

$$K^- = \left\{t: v_0(t) \leqslant -1 + \frac{\delta}{3} < -1 + \frac{2\delta}{3} \leqslant v_1(t)\right\}.$$

Let V^+, V^- be, respectively, disjoint neighborhoods of K^+, K^-. Note that K^+, K^- are compact G_δ's, $K^+ \supset \text{crit}\, v_0^+$, and $K^- \supset \text{crit}\, v_0^-$. By Urysohn's lemma, we can choose a function $f \in C_0(T)$ such that

$$f = \begin{cases} 1/2 & \text{on } K^+ \\ -1/2 & \text{on } K^-, \\ 0 & \text{off } V^+ \cup V^-, \end{cases}$$

and $|f| < \frac{1}{2}$ off $K^+ \cup K^-$. Set $x = v_0 - f$. Then $\| x - v_0 \| = \frac{1}{2}$,

$$\operatorname{crit}(v_0 - x)^+ = K^+,$$

and $\operatorname{crit}(v_0 - x)^- = K^-$. Since $v_1 < v_0$ on K^+, and $v_1 > v_0$ on K^-, $v_1 \in K(v_0, x)$. Let $J = \{t : |v_0(t)| \geqslant 1/2\}$. Since $\operatorname{crit} v_0 \subset \operatorname{int}(K^+ \cup K^-)$ and $J \sim \operatorname{int}(K^+ \cup K^-)$ is compact ("int" means "interior of"), it follows that

$$\sup\{|v_0(t)| : t \in J \sim \operatorname{int}(K^+ \cup K^-)\} = 1 - \delta_1$$

for some $\delta_1 > 0$. Set $\epsilon = \min\{\delta/6, \delta_1/2\}$. Let $v \in B(v_0, \epsilon) \cap K(v_0, x)$, i.e., $\| v_0 - v \| < \epsilon$, $v < v_0$ on K^+, and $v > v_0$ on K^-. In particular, $|v| < 1$ on $K^+ \cup K^-$. If $t \in J \sim K^+ \cup K^-$, then

$$|v(t)| < |v_0(t)| + \epsilon \leqslant 1 - \delta_1 + \epsilon < 1.$$

If $t \notin J$, then

$$|v(t)| < |v_0(t)| + \epsilon < 1/2 + \epsilon \leqslant 1.$$

Thus $\| v \| < 1$ and so, $B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$, i.e., v_0 is strongly non-lunar. ∎

From Theorems 2.18 and 4.1, we immediately obtain

COROLLARY 4.2. *In $C_0(T)$, a set is a sun if and only if it is a moon.*

LEMMA 4.3. *Let $v_0 \in S(C_0(T))$. Then v_0 is QP if and only if* crit v_0 *is clopen* (i.e., *both open and closed*).

Proof. Let v_0 be *QP*. Choose an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1).$$

Suppose $\sup\{|v_0(t)| : t \notin \operatorname{crit} v_0\} = 1$. Without loss of generality, we may assume $\sup\{v_0(t) : t \notin \operatorname{crit} v_0\} = 1$. Choose $t_0 \in T \sim \operatorname{crit} v_0$ such that

$$v_0(t_0) > 1 - \epsilon/2.$$

Using Urysohn's lemma, choose an $x \in C_0(T)$ such that

$$x = \begin{cases} -\epsilon/2 & \text{on } \operatorname{crit} v_0^+, \\ \epsilon/2 & \text{on } \{t_0\} \cup \operatorname{crit} v_0^-, \end{cases}$$

and $|x| \leqslant \epsilon/2$ everywhere. Setting $v = v_0 + x$, we see that $\| v - v_0 \| < \epsilon$ and $|v(t)| < 1$ on crit v_0, i.e., $v \in B(v_0, \epsilon) \cap K(v_0, 0)$. But

$$v(t_0) = v_0(t_0) + \epsilon/2 > 1, \text{ so } \| v \| > 1.$$

This contradiction shows that

$$\sup\{| v_0(t)| : t \notin \text{crit } v_0\} < 1,$$

i.e., there exists $\delta > 0$ such that

$$\text{crit } v_0 = \{t \in T : | v_0(t)| > 1 - \delta\}.$$

Hence crit v_0 is open. Also, crit v_0 is always closed.

Conversely, suppose crit v_0 is open. Then there is $\delta > 0$ such that

$$\text{crit } v_0 = \{t : | v_0(t)| > 1 - \delta\}.$$

Let $\epsilon = \delta/2$. If $v \in X$, $\| v - v_0 \| < \epsilon$, and $| v(t)| < 1$ on crit v_0, then for any $t \in T \sim \text{crit } v_0$, we have

$$| v(t)| < | v_0(t)| + \epsilon \leqslant 1 - \delta + \epsilon < 1$$

and so, $\| v \| < 1$. We have shown that $B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1)$ and so, v_0 is QP. ▮

THEOREM 4.4. *The following are equivalent*:

(1) $C_0(T)$ *is a QP-space.*

(2) crit v_0 *is clopen for every* $v_0 \in C_0(T)$.

(3) T *is discrete.*

Proof. The equivalence of (1) and (2) is an immediate consequence of Lemma 4.3.

(3) ⇒ (2). If T is discrete, then every subset of T is clopen.

(2) ⇒ (3). Suppose crit v_0 is open for every $v_0 \in X$. If $T_0 \subset T$ is compact, then every continuous function on T_0 must have a finite range. Using the regularity of T_0, it would then follow that T_0 is finite. Hence compact sets are finite; so T is discrete. ▮

5. THE SPACE $L_1(T, \Sigma, \mu)$

In this section, unless otherwise specified, (T, Σ, μ) will denote a σ-finite measure space and $X = L_1 = L_1(T, \Sigma, \mu)$ the space of all real-valued integrable functions x on T, endowed with the norm

$$\| x \| = \int_T | x(t)| \, d\mu.$$

We shall abbreviate "μ-almost everywhere" to "a.e." The *zero set* of a given measurable function x is defined, modulo a set of measure zero, by

$$Z(x) = \{t \in T : x(t) = 0\}.$$

The *support of* x is defined by

$$\operatorname{supp} x = T \sim Z(x) = \{t : x(t) \neq 0\}.$$

A set $A \in \Sigma$ is called an *atom* if $0 < \mu(A) < \infty$ and each measurable subset $B \subset A$ satisfies either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. It is well known (and easy to prove) that (T, Σ, μ) can have at most countably many atoms. A subset of T is called *purely atomic* if it is the union of atoms. Each measurable function x must be constant a.e. on an atom A. We denote this value by $x(A)$.

LEMMA 5.1. *Let* $v_0 \in S(L_1)$. *Then*

$$\mathfrak{A}(v_0) = \{x \in X : | x | \leqslant | v_0 | \text{ a.e., } \textit{and} \operatorname{sgn} x = \operatorname{sgn} v_0 \text{ a.e. } \textit{on} \operatorname{supp} x\}.$$

Proof. We have $x \in \mathfrak{A}(v_0)$ iff $\| v_0 - x \| + \| x \| = \| v_0 \|$. By the condition for equality in the triangle inequality [18, p. 192], this is equivalent to the existence of a positive measurable function ρ such that

$$v_0 = (1 + \rho)\, x \text{ a.e., on } \operatorname{supp}[(v_0 - x)\, x]. \tag{*}$$

But (*) is clearly equivalent to $| x | \leqslant | v_0 |$ a.e. and $\operatorname{sgn} x = \operatorname{sgn} v_0$ a.e. on $\operatorname{supp} x$. ▮

The following result is the main tool of this section.

LEMMA 5.2. *Let* $v_0 \in S(L_1)$. *Consider the statements*:

(1) $\operatorname{supp} v_0$ *is purely atomic,*

(2) v_0 *is strongly nonlunar,*

(3) v_0 *is nonlunar,*

(4) $\operatorname{supp} v_0$ *contains an atom.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2). Let $\operatorname{supp} v_0 = \bigcup_{i \in I} A_i$, where the A_i are atoms and I is some (countable) index set. Let $v_1 \in K(v_0, 0)$, i.e.

$$\int_{v_0>0} (v_1 - v_0)\, d\mu - \int_{v_0<0} (v_1 - v_0)\, d\mu + \int_{Z(v_0)} | v_1 - v_0 |\, d\mu < 0.$$

By a limit argument (using e.g., the dominated convergence theorem) one can readily show that there exists $\delta > 0$ such that

$$\sum_{I_\delta^+} [v_1(A_i) - v_0(A_i)]\, \mu(A_i) - \sum_{I_\delta^-} [v_1(A_i) - v_0(A_i)]\, \mu(A_i)$$
$$+ \int_{Z(v_0)} | v_1 |\, d\mu + \sum_{I \sim I_\delta^+ \cup I_\delta^-} | v_1(A_i) - v_0(A_i)|\, \mu(A_i) < 0,$$

where $I_\delta^+ = \{i \in I : v_0(A_i)\, \mu(A_i) > \delta\}$, and $I_\delta^- = \{i \in I : v_0(A_i)\, \mu(A_i) < -\delta\}$. Define a function x by

$$x(t) = \begin{cases} v_0(A_i), & \text{if} \quad t \in A_i \quad \text{and} \quad i \in I \sim I_\delta^+ \cup I_\delta^-, \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{O}\!\mathcal{l}(v_0)$ and

$$K(v_0, x) = \Big\{ v \in X: \int_{v_0 > x} (v - v_0)\, d\mu - \int_{v_0 < x} (v - v_0)\, d\mu$$
$$+ \int_{v_0 = x} | v - v_0 |\, d\mu < 0 \Big\}$$
$$= \Big\{ v \in X: \sum_{I_\delta^+} [v(A_i) - v_0(A_i)]\, \mu(A_i) - \sum_{I_\delta^-} [v(A_i) - v_0(A_i)]\, \mu(A_i)$$
$$+ \int_{Z(v_0)} | v |\, d\mu + \sum_{I \sim I_\delta^+ \cup I_\delta^-} | v(A_i) - v_0(A_i)|\, \mu(A_i) < 0 \Big\}.$$

In particular, $v_1 \in K(v_0, x)$. Choose any $0 < \epsilon < \delta$. Let

$$v \in B(v_0, \epsilon) \cap K(v_0, x).$$

Then

$$| v_0(A_i) - v(A_i)|\, \mu(A_i) < \epsilon < \delta, \qquad \text{for all} \quad i \in I,$$

so that $\operatorname{sgn} v(A_i) = \operatorname{sgn} v_0(A_i)$ if $i \in I_\delta^+ \cup I_\delta^-$. Thus

$$\| v \| - 1 = \| v \| - \| v_0 \|$$
$$= \sum_{I_\delta^+} [v(A_i) - v_0(A_i)]\, \mu(A_i) - \sum_{I_\delta^-} [v(A_i) - v_0(A_i)]\, \mu(A_i)$$
$$+ \sum_{I \sim I_\delta^+ \cup I_\delta^-} [| v(A_i)| - | v_0(A_i)|]\, \mu(A_i) + \int_{Z(v_0)} | v |\, d\mu$$
$$\leqslant \sum_{I_\delta^+} [v(A_i) - v_0(A_i)]\, \mu(A_i) - \sum_{I_\delta^-} [v(A_i) - v_0(A_i)]\, \mu(A_i)$$
$$+ \sum_{I \sim I_\delta^+ \cup I_\delta^-} | v(A_i) - v_0(A_i)|\, \mu(A_i) + \int_{Z(v_0)} | v |\, d\mu < 0,$$

since $v \in K(v_0, x)$. Hence

$$B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$$

and so, v_0 is strongly nonlunar.

The implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4). If v_0 is nonlunar, then there is $x \in \mathcal{O}\!\mathcal{l}(v_0)$ and $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1).$$

By Lemma 5.1, for almost all $t \in T$, either $0 \leqslant x(t) \leqslant v_0(t)$ or $v_0(t) \leqslant x(t) \leqslant 0$. Now

$$K(v_0, x) = \left\{v \in L_1 : \int_{v_0>x} (v - v_0)\, d\mu - \int_{v_0<x} (v - v_0)\, d\mu + \int_{v_0=x} | v - v_0 |\, d\mu < 0\right\}.$$

Let $T^+ = \{t \in \operatorname{supp} v_0 : \ v_0(t) > x(t)\}$ and $T^- = \{t \in \operatorname{supp} v_0 : v_0(t) < x(t)\}$. It follows that either $\mu(T^+) > 0$ or $\mu(T^-) > 0$. We may assume $\mu(T^+) > 0$; the case $\mu(T^-) > 0$ can be treated similarly. If supp v_0 contained no atom, then neither would T^+. Hence we can choose a sequence (E_n) of disjoint subsets of T^+ with $0 < \mu(E_n) < \infty$. Since

$$\sum_1^\infty \int_{E_n} | v_0 |\, d\mu \leqslant \int_T | v_0 |\, d\mu = 1,$$

we have $\int_{E_N} | v_0 |\, d\mu \to 0$. Choose N such that $\int_{E_N} v_0\, d\mu < \epsilon/4$, and let $E = \bigcup_1^\infty E_n$. Define a function v by

$$v = \begin{cases} v_0 & \text{on } T \sim E, \\ (1 + \delta)\, v_0 & \text{on } E \sim E_N, \\ -2v_0 & \text{on } E_N, \end{cases}$$

where

$$\delta = \left[\int_{E\sim E_N} v_0\, d\mu\right]^{-1} \int_{E_N} v_0\, d\mu.$$

Then

$$\int_{v_0>x} (v - v_0)\, d\mu - \int_{v_0<x} (v - v_0)\, d\mu + \int_{v_0=x} | v - v_0 |\, d\mu$$
$$= \delta \int_{E\sim E_N} v_0\, d\mu - 3 \int_{E_N} v_0\, d\mu = -2 \int_{E_N} v_0\, d\mu < 0,$$

i.e., $v \in K(v_0, x)$,

$$\| v - v_0 \| = \delta \int_{E \sim E_N} | v_0 | \, d\mu + 3 \int_{E_N} | v_0 | \, d\mu = 4 \int_{E_N} | v_0 | \, d\mu < \epsilon,$$

i.e., $v \in B(v_0, \epsilon)$, but

$$\| v \| = \int_{T \sim E} | v_0 | \, d\mu + (1 + \delta) \int_{E \sim E_N} | v_0 | \, d\mu + 2 \int_{E_N} | v_0 | \, d\mu$$

$$= 1 + \delta \int_{E \sim E_N} | v_0 | \, d\mu + \int_{E_N} | v_0 | \, d\mu > 1.$$

However, this contradicts

$$B(v_0, \epsilon) \cap K(v_0, x) \subset B(0, 1)$$

and completes the proof. ∎

From this result we immediately obtain

COROLLARY 5.3. *If* supp v_0 *contains no atom, then* v_0 *is a lunar point of* $S(L_1)$. *In particular, if* T *contains no atoms,* $S(L_1)$ *is a moon.*

Another easy consequence of Lemma 5.2 is

THEOREM 5.4. *The following are equivalent*:

(1) $L_1(T, \Sigma, \mu)$ *is strongly nonlunar,*

(2) *each point of* $S(L_1)$ *is nonlunar,*

(3) T *is purely atomic,*

(4) $L_1(T, \Sigma, \mu)$ *is (isometrically isomorphic to) a space of type* l_1 *or* l_1^n, *for some* n.

Proof. The implication (1) ⇒ (2) is obvious.

(2) ⇒ (3). If T were not purely atomic, there would exist a set $E \in \Sigma$, with $0 < \mu(E) < \infty$, containing no atoms. Then the support of the element $v_0 = [\mu(E)]^{-1} \chi_E$ would contain no atom. By Lemma 5.2, v_0 would be lunar.

The equivalence (3) ⇔ (4) is well known.

(3) ⇒ (1). Since T is purely atomic, so is supp v_0 for every $v_0 \in S(L_1)$. By Lemma 5.2, it follows that $L_1(T, \Sigma, \mu)$ is strongly nonlunar. ∎

LEMMA 5.5. *Let* $v_0 \in S(L_1)$. *Then* v_0 *is a QP point if and only if* supp v_0 *is a finite union of atoms.*

Proof. Let v_0 be QP. Then there is an $\epsilon > 0$ such that

$$B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1).$$

Let

$$T^+ = \{t \in T : v_0(t) > 0\}, \qquad T^- = \{t \in T : v_0(t) < 0\}.$$

If supp v_0 were not a finite union of atoms, then supp v_0 would contain either an infinite number of atoms or a set of positive measure which has no atoms. In either case, one of the sets T^+ or T^- would contain a sequence (E_n) of disjoint sets with $0 < \mu(E_n) < \infty$. We may assume it is T^+ as the other possibility can be treated similarly. The proof now proceeds exactly as that of the implication (3) $\Rightarrow$ (4) in Lemma 5.2 (taking $x = 0$). Thus we can construct a function $v \in B(v_0, \epsilon) \cap K(v_0, 0)$ with $\| v \| > 1$ and get a contradiction.

Conversely, suppose supp $v_0 = \bigcup_{i=1}^n A_i$, where each A_i is an atom; we can assume $\mu(A_i \cap A_j) = 0$ if $i \neq j$. Choose $\epsilon > 0$ such that

$$\epsilon < \tfrac{1}{2} \min_{1 \leqslant i \leqslant n} | v_0(A_i)| \, \mu(A_i).$$

Let $v \in B(v_0, \epsilon)$. Then sgn $v(A_i) = sgn\, v_0(A_i)$ for $i = 1,..., n$. If v is also in $K(v_0, 0)$, then

$$\begin{aligned} \| v \| - 1 &= \| v \| - \| v_0 \| \\ &= \int_{v_0 > 0} (v - v_0)\, d\mu - \int_{v_0 < 0} (v - v_0)\, d\mu + \int_{Z(v_0)} | v |\, d\mu < 0, \end{aligned}$$

i.e., $\| v \| < 1$. Hence $B(v_0, \epsilon) \cap K(v_0, 0) \subset B(0, 1)$ and so, v_0 is QP. ∎

From this lemma we immediately obtain

THEOREM 5.6. *The following are equivalent*:

(1) *$L_1(T, \Sigma, \mu)$ is a QP-space.*

(2) *T is a finite union of atoms.*

(3) *$L_1(T, \Sigma, \mu)$ is (isometrically isomorphic to) a space of type l_1^n for some n.*

6. Related Matters and Some Open Questions

Let T be a compact Hausdorff space, X a real normed linear space, and let $C(T, X)$ be the normed linear space of all X-valued continuous functions f on T, with the max norm : $\| f \| = \max_{t \in T} \| f(t) \|_X$. If T is a singleton, $C(T, X) = X$; while if $X = R$, $C(T, X) = C(T)$. It is natural to ask questions like "if X has a certain property, does $C(T, X)$ too have this property?" In particular, the following problem is unsettled:

Problem 6.1. If X is strongly nonlunar (or even QP), is $C(T, X)$ strongly nonlunar?

We do have the following partial answer to this question: *If T is finite, then $C(T, X)$ is strongly nonlunar (resp. QP), if X is strongly nonlunar (resp. QP).* This fact is a consequence of Theorem 3.2, since we may regard $C(T, X)$ as $(\prod_{t \in T} X_t)_{c_0(T)}$, where $X_t = X$ for every $t \in T$.

Another open question is whether the converse of Theorem 2.18 holds. Thus

Problem 6.2. If each moon in X is a sun, must X be strongly nonlunar?

We have seen that there are strongly nonlunar spaces which are not QP. However, we know of no finite-dimensional example.

In [12], Brown introduced the concept of a normed linear space having property (P). (X has property (P) if for each pair of points x, z in X, with $\| x + z \| \leqslant \| x \|$, there are positive constants λ, ϵ such that $\| y + \lambda z \| \leqslant \| y \|$ whenever $\| x - y \| < \epsilon$.) Brown observed that every strictly convex space has (P), and so does every finite-dimensional space whose unit ball is a convex polytope. He also showed that *a space X has (P) if and only if the metric projection onto any finite-dimensional subspace of X is lower semicontinuous* (cf. also [2].) Blatter, Morris and Wulbert [2] have shown that $C_0(T)$ has property (P) if and only if T is discrete. Also, they verified that $L_1(T, \Sigma, \mu)$ has property (P) if and only if T is a finite union of atoms. In [1] Blatter proved, among other things, that $(\prod_{i \in I} X_i)_{c_0(I)}$ has property (P) if and only if each of the spaces X_i has (P). Thus, in the spaces $C_0(T)$, $L_1(T, \Sigma, \mu)$ and $(\prod_{i \in I} X_i)_{c_0(I)}$, property (P) is equivalent to QP.

Deutsch and Lindahl [15] have studied the minimal extremal subsets of the unit sphere. Let $v_0 \in S(X)$ and let $E(v_0)$ denote the minimal extremal subset of $S(X)$ which contains v_0. Then X is said to have property Q if, for each $v_0 \in S(X)$, the set $E(v_0)$ is the intersection of all the exposed sets in $S(X)$ which contain v_0. It was shown in [15] that $C_0(T)$ has property Q if and only if T is discrete; $L_1(T, \Sigma, \mu)$ has property Q if and only if T is a finite union of atoms; every finite-dimensional space whose unit ball is a polytope has property Q.

Thus, from the preceding two paragraphs, we have

THEOREM 6.3. *Let $X = C_0(T)$ or $X = L_1(T, \Sigma, \mu)$. Then the following are equivalent*:

(1) *X is QP.*

(2) *X has property (P) (of [12]).*

(3) *X has property Q (of [15]).*

If $X = C_0(T)$, each of these conditions is equivalent to T being discrete. If $X = L_1(T, \Sigma, \mu)$, each of these conditions is equivalent to T being a finite union of atoms.

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References

1. J. Blatter, Zur stetigen Abhängigkeit der Menge der besten Approximierenden eines Elementes in einem normierten reellen Vektorraum, *in* "Numerische Mathematik, Differentialgleichungen, Approximationstheorie," (L. Collatz and G. Meinardus, Eds.), ISNM Vol. 9, pp. 225–227, Birkhauser, Basel, 1968.
2. J. Blatter, P. Morris, and D. Wulbert, Continuity of the set-valued metric projection, *Math. Annal.* **178** (1968), 12–24.
3. W. Breckner and I. Kolumban, Théorèmes de caractérisation des éléments de la meilleure approximation, *C. R. Acad. Sci. Paris*, **226** (1968), 206–208.
4. W. Breckner and I. Kolumban, Über die Charakterisierung von Minimallösungen in linearen normierten Räumen, *Mathematica (Cluj)* **10** (1968), 33–46.
5. B. Brosowski, Nicht-lineare Tschebyscheff-Approximation, B. I.-Hochschulskripten, 808/808a, Mannheim, 1968.
6. B. Brosowski, Nichtlineare Approximation in normierten Vektorräumen, *in* "Abstract Spaces and Approximation" (P. L. Butzer and B. Sz.-Nagy, Eds.) ISNM Vol. 10, Birkhäuser, Basel, 1969.
7. B. Brosowski and H. Weber, Zum Begriff der Regularität in normierten Vektorräumen, Max-Planck-Institut Technical Report, München, March 1969.
8. B. Brosowski, Einige Bemerkungen zum verallgemeinerten Kolmogoroffschen Kriterium, *in* "Funktionalanalytische Methoden der Numerische Mathematik," (L. Collatz and H. Unger, Eds.), ISNM Vol. 12, Birkhäuser, Basel, 1969.
9. B. Brosowski and F. Deutsch, Characterization of Kolmogorov sets in $C(Q, l_1^n)$, Aug. 1969, unpublished.
10. B. Browowski and R. Wegmann, Charakterisierung von besten Approximationen in normierten Vektorräumen, *J. Approximation Theory*, to appear.
11. B. Brosowski and F. Deutsch, On some geometrical properties of suns, *J. Approximation Theory*, to appear.
12. A. L. Brown, Best *n*-dimensional approximation to sets of functions, *Proc. London Math. Soc.*, **14** (1964), 577–594.
13. M. M. Day, "Normed Linear Spaces," Springer-Verlag, New York, 1958.
14. F. R. Deutsch and P. H. Maserick, Applications of the Hahn-Banach theorem in approximation theory, *SIAM Rev.* **9** (1967), 516–530.
15. F. Deutsch and R. Lindahl, Minimal extremal subsets of the unit sphere, *Math. Annalen*, to appear.
16. N. Dunford and J. T. Schwartz, "Linear Operators," Vol. I, Interscience, New York, 1958.

17. N. Efimov and S. Steckin, Some properties of Chebyshev sets, *Dokl. Akad. Nauk SSSR* **118** (1958), 17–19 (Russian).
18. E. Hewitt and K. Stromberg, "Real and Abstract Analysis," Springer-Verlag, New York, 1965.
18a. V. Klee, Some characterizations of convex polyhedra, *Acta Math.* **102** (1959), 79–107.
19. P. H. Maserick, Convex polytopes in linear spaces, *Illinois J. Math.* **9** (1965), 623–635.
20. I. Singer, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer–Verlag, New York, 1970.
21. L. Vlasov, On Chebyshev sets in Banach spaces, *Dokl. Akad. Nauk SSSR* **141** (1961), 19–20 (Russian).